Lecture 6: D'Alembert's Wave Equation.

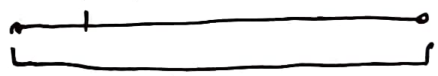
Basic Case: Vibrating String

- Imagine a string on a harp: 

The string is held at both ends at a certain length ℓ under tension T to meet a pitch. The string itself matters to the pitch too; consider its linear density ρ.

T: Newtons or pounds, etc
ℓ: feet, meters, etc.
ρ: mass/length (g/m)

~Assume that the displacement of the string (perpendicular to itself) is small compared to its length, so that we can assume T is constant. Further, ignore gravity and other forces, assume ρ is fixed along the string (the string is homogeneous)

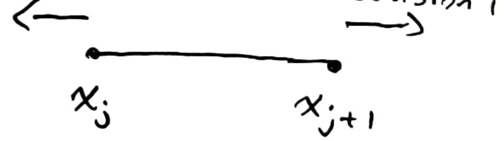


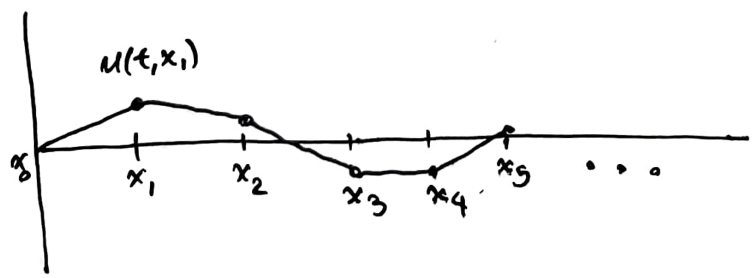
- Parameterize the string by $x \in [0, \ell]$ such that we can model the displacement by $u(t, x): [0, \infty)_t \times [0, \ell]_x \to \mathbb{R}$.

We discretize our model: divide the length into n pieces of size $\Delta x = \ell/n$ for some large n.

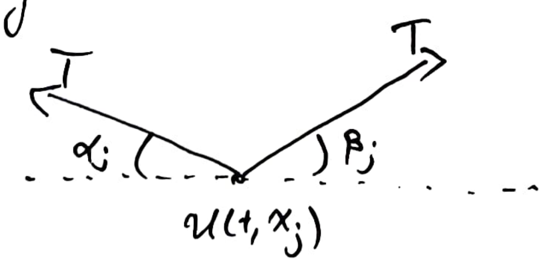
Discretization is often used for computers to solve numerical problems

- Each segment has mass $\rho \cdot \Delta x$
- For $j = 0, 1, \dots n$ let $x_j = j\Delta x$





Force diagram:



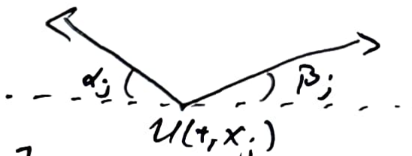
- Vertical force at $u(t, x_j)$ is
$$\Delta F(t, x_j) = T\sin(\alpha_j) + T\sin(\beta_j)$$

- Our displacement being small means $\underbrace{\alpha_j, \beta_j << 1}_{\text{are much less than } 1}$

this allows us to approximate $\cos(\alpha_j) \approx 1$

$$\sin(\alpha_j) \approx \tan(\alpha_j) = \frac{u(t, x_{j-1}) - u(t, x_j)}{\Delta x} \qquad \left(\frac{\text{opposite}}{\text{adjacent}} \right)$$

$$\sin(\beta_j) \approx \frac{u(t, x_{j+1}) - u(t, x_j)}{\Delta x}$$



Hence, $\Delta F(t, x_j) = \frac{T}{\Delta x}\left[u(t, x_{j+1}) + u(t, x_{j-1}) - 2u(t, x_j) \right]$

under the approximation.

- Next, Newton's law says Force = mass x accelleration

or $\Delta F(t, x_j) = (\rho \Delta x)\left(\frac{\partial^2 u}{\partial t^2}(t, x_j) \right)$

By the approx., this is $= \frac{T}{\Delta x}\left[u(t, x_{j+1}) + u(t, x_{j-1}) - 2u(t, x_j) \right]$

So $\frac{\partial^2 u}{\partial t^2}(t, x_j) = \frac{T}{\rho (\Delta x)^2}\left[u(t, x_{j+1}) + u(t, x_{j-1}) - 2u(t, x_j) \right]$

- Now, to un-discretize, we take $n \to \infty$, or $\Delta x \to 0$.

Assume $u \in C^2$ so that

$$\lim_{\Delta x \to 0} \frac{u(t, x_{j+1}) - 2u(t, x_j) + u(t, x_{j-1})}{(\Delta x)^2} = \frac{\partial^2 u}{\partial x^2}(t, x)$$

$\to$ If this is unfamiliar, use a Taylor Series or L'Hopital's Rule.

- Thus, as $\Delta x \to 0$, we have the PDE

(J) $\frac{\partial^2 u}{\partial t^2} - \frac{T}{\rho} \frac{\partial^2 u}{\partial x^2} = 0$ the Wave Equation.

with boundary conditions $u(t, 0) = u(t, \ell) = 0$

- In the wild, you may also see $\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$ or

$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

$\hookrightarrow$ c: wave speed or propagation speed.

We will use the latter

(K) $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

and add boundary conditions later in our analysis.

Wave Equation Characteristics

- We use a trick of "factoring" our differential operator (which works on C^2 functions)

$$\left(\frac{\partial^2}{\partial t^2} - c\frac{\partial^2}{\partial x^2}\right)u = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = 0$$

$\to$ $\frac{\partial}{\partial t} \pm c\frac{\partial}{\partial x}$, as a differential operator, gives PDE $\frac{\partial u}{\partial t} \pm c\frac{\partial u}{\partial x} = 0$ with characteristic $\begin{cases} \dot{x} = \pm c \\ x(0) = x_0 \end{cases}$ or $x(t) = x_0 \pm tc$.

we use these to construct a solution

$\boxed{\text{Th}^{\underline{m}}}$ Under the initial conditions $u(0,x) = g(x)$, $\frac{\partial u}{\partial t}(0,x) = h(x)$ for $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, the wave equation (K) admits a unique solution

$$u(t,x) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\,d\tau \qquad (L)$$

$\boxed{\text{Pf}}$ Define $w(t,x) = \frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}$. Then, using the "factoring" above, $\frac{\partial w}{\partial t} + c\frac{\partial w}{\partial x} = 0$ (a linear conservation equation)

Since the characteristics are given by $x_+(t) = x_0 + ct$, we have for initial conditions $w(0,x) = w_0(x)$ a solution by the method of characteristics
$w(t,x) = w_0(x-ct)$ that is unique (why?).

Next, we know w as a function (once we compute w_0), so $\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x} = w$, the definition of w, gives another conservation equation, with characteristic equations $x_-(t) = x_0 - ct$. Thus, the method of characteristics gives

$$\frac{d}{dt} u(t, x_0 - ct) = w(t, x_0 - ct)$$

with initial condition $u(0,x) = g(x)$, we have the unique solution

$$u(t, x_0 - ct) = g(x_0) + \int_0^t w(s, x_0 - cs)\,ds$$

or

$$u(t,x) = g(x+ct) + \int_0^t w(s, x - c(s-t))\,ds$$

Applying our formula for w

$$u(t,x) = g(x+ct) + \int_0^t w_0(x - 2cs + ct)\,ds$$

To simplify the integrand, set $\tau = x - 2cs + ct$ so

$$u(t,x) = g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} w_0(\tau)\,d\tau$$

We compute w_0 to finish

$$w_0(x) = \frac{\partial u}{\partial t}(0,x) - c\frac{\partial u}{\partial x}(0,x) = h(x) - c\frac{\partial g}{\partial x}(x)$$

and

$$u(t,x) = g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} -c\frac{\partial g}{\partial x}(\tau)\,d\tau + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\,d\tau$$

$$= \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\,d\tau \qquad \square$$

- Recall that c was called the "propagation speed". To see this, let $u_\pm(x) = \frac{1}{2}g(x) \mp \frac{1}{2c}\int_0^x h(\tau)\,d\tau$

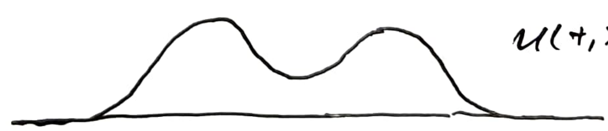
  so $u(t,x) = u_+(x-ct) + u_-(x+ct)$

  then, u_+ propagates to the right and u_- to the left with speed controlled by the parameter c.

ex.) Consider $h(x) = 0$, $g(x) = \begin{cases} (1-x^2)^2 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}$ initial conditions for the wave equation. The solution is sketched below

$t = 0$ $u(t,x)$

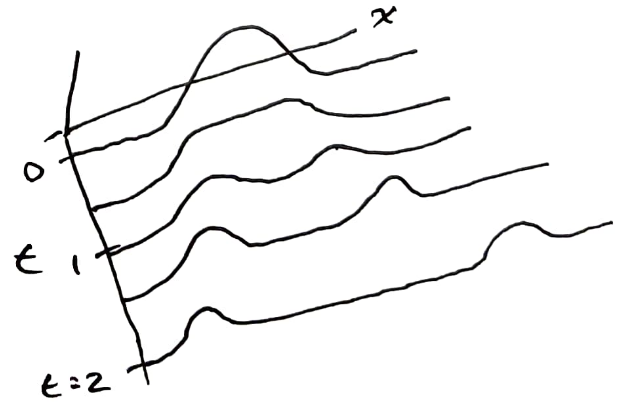
$t = 1$ $\xleftarrow{u_-}$ $\xrightarrow{u_+}$



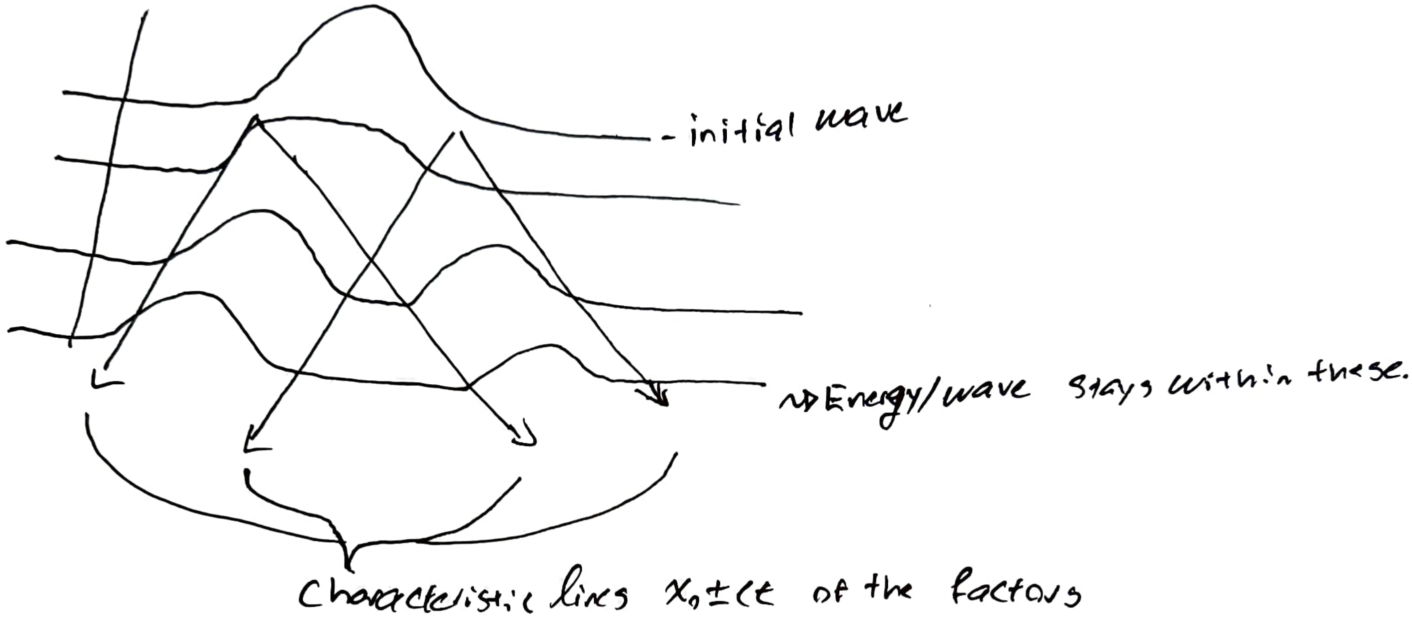
$u(t,x)$

$t = 2$

$u(t,x)$



- Notice the V-pattern of movement: the waves move within the span of characteristic lines:



- This is related to a law on wave propagation:

<u>Huygen's Principle in One Dimension</u>

Suppose u solves the wave equation for $t \geq 0$, $x \in \mathbb{R}$, with $u(0,x) = g(x)$, $\frac{\partial u}{\partial t}(0,x) = h(x)$ for g, h supported in the bounded interval $[a,b]$. Then,

$$\text{supp}(u) \subset \{(t,x) \in \mathbb{R}^+ \times \mathbb{R} \,;\, x \in [a-ct, b+ct]\}$$

$\boxed{\text{Pf}}$ Since $u(t,x) = \frac{1}{2}[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\tau)\,d\tau$,

the terms of g vanish unless $x+ct, x-ct \in [a,b]$ or

$a+ct \leq x \leq b+ct$ for each.
$a-ct \leq x \leq b-ct$

The integral on h vanishes unless $[x-ct, x+ct]$ overlaps $[a,b]$, i.e. $x \in [a-ct, b+ct]$. $\square$